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Surface tension effect in the two-fluids equation system

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Abstract

The difficulty of pressure discontinuity in the two-fluids formulation, caused by surface tension at the interface, can now be resolved by a new concept we call 'surface tension thickness'. It removes one of the major barriers that the conventional two-phase flow formulation has elicited: the ill-posedness of the differential equation system. The three sets of real eigenvalues we have found at the present formulation represent such existing two-phase flow regimes as the homogeneous, slug, and separated flows. The pressure wave propagation speeds in the two-phase flows predicted by the present formulation show good agreement with the experimental data.

Nomenclature

a interfacial area intensity
A interfacial area
A, **B** coefficient matrices
c constant
C speed of sound
E source vector
H state vectors
I identity matrix
L bulk modulus
p pressure
R radius of bubble
t time
v velocity
V control volume
x space coordinate.

Greek symbols
α void fraction
δ surface tension thickness
λ eigenvalue of a matrix
ρ density
σ surface tension.

Subscripts
i interface
k index for each fluid

Superscript
-1 inverse of matrix.

1. Introduction

It has been well known that the majority of the differential equation systems governing the two-fluids flow constitute an ill-posed initial value problem due to their complex characteristics [1, 2]. Researchers have attempted in the past without much success to render hyperbolic property to the equation systems by modifying some of the existing terms. In the ref. [3], derivation of the two-fluids equation is given.

In the present paper, the pressure discontinuity at the two-fluids interface is adequately treated by a new concept called 'surface tension thickness'. Taking the change of interfacial area into account in the momentum equations, we have been able to show that the equation system has real eigenvalues that depend upon the effective bulk moduli associated with the surface tension. Such two-fluids flows as the homogeneous, slug, and separated flows are the examples having real eigenvalues in the

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present formulation. We will show that the pressure-wave propagation speeds represented by these real eigenvalues are very physical and comparable with the experimental data. The pattern of characteristic roots in the parameter plane shows, for very small surface tension, fractal-like features much observed in chaos.

2. Classification of pressure discontinuity

The basic two-fluids equation system, based on macroscopic relationship among the instantaneous area-averaged values for all state and flow parameters, takes the following form for the inviscid and one-dimensional flow:

$$\frac{\partial(\alpha_k\rho_k)}{\partial t}+\frac{\partial(\alpha_k\rho_k v_k)}{\partial x}=0 \quad (1)$$

$$\frac{\partial(\alpha_k\rho_k v_k)}{\partial t}+\frac{\partial(\alpha_k\rho_k v_k^2)}{\partial x}+\alpha_k\frac{\partial p_k}{\partial x}+(p_k-p_i)\frac{\partial\alpha_k}{\partial x}=0 \quad (2)$$

where $\alpha_1+\alpha_2=1$, and $k=1,2$.

Young and Laplace [4] gave the fundamental equation of surface tension by $p_2-p_1=2\sigma/R_i$ for a sphere of radius R_i. Since it is known that the ill-posedness of the equation system is caused by the ill-treatment of the pressure discontinuity at the interface, we reformulate Young and Laplace's equation using the concept of surface tension thickness. We assume here that the film thickness δ between the inner radius R_2 and outer radius R_1 is very small as shown in Fig. 1. Then, we can write

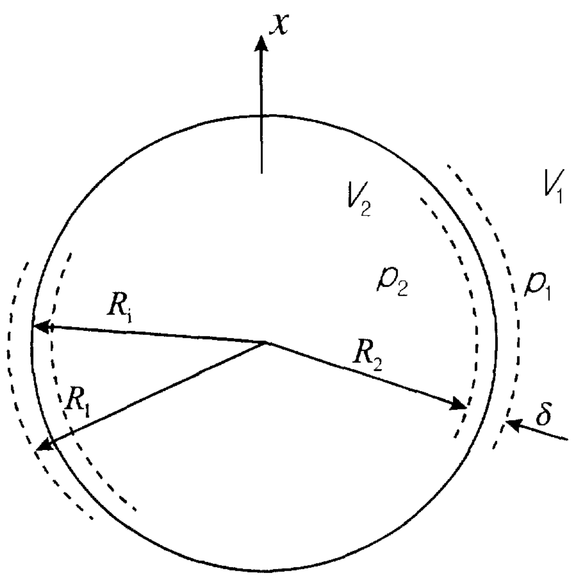

Fig. 1. Thin film of thickness δ separates the inner sphere of radius R_2 (volume V_2) and the outer sphere of radius R_1 (volume V_1). The midsphere has radius $R_i=(R_2+R_1)/2$ and surface area A_i.

$$p_2-p_1=\frac{2\delta}{R_2+\delta/2}\left(\frac{\sigma}{\delta}\right)=\frac{2\delta}{R_1-\delta/2}\left(\frac{\sigma}{\delta}\right). \quad (3)$$

Here R_i represents either $R_2+\delta/2$ or $R_1-\delta/2$, namely, the distance from the sphere center of the midplane of the thin film. The surface tension thickness δ then can be regarded as a hypothetical interfacial thickness stated earlier in the statistical mechanics. The surface increment of the interfacial sphere, ΔA_i, caused by the increment of its radius, ΔR_i, can be correlated with the increment of the inner and outer sphere volumes, ΔV_1 and ΔV_2, by

$$\frac{R_2}{2}\left(\frac{\Delta A_i/V}{\Delta V_2/V}\right)=1-\frac{\delta/2}{R_2+\delta/2} \quad (4)$$

$$\frac{R_1}{2}\left(\frac{\Delta A_i/V}{\Delta V_1/V}\right)=-1-\frac{\delta/2}{R_1-\delta/2} \quad (5)$$

where V is V_1+V_2. We now assume that the pressure discontinuity has phasic components as

$$p_2-p_1=(p_2-p_i)+(p_i-p_1) \quad (6)$$

where p_i is a hypothetical interfacial pressure. Likely, we split the term σ/δ into the plastic components of bulk moduli based on the concept of surface tension thickness [5–7]:

$$\frac{4\sigma}{\delta}=L_1+L_2. \quad (7)$$

The term L_1 and L_2, corresponding to the conventional Lagrangian multipliers [8], play a very important role by distinguishing the two fluids in the system characteristics.

For the limiting case, $\Delta R_i\to 0$, we can reduce equation (4) multiplied by a factor $\partial\alpha_2/\partial x$ and equation (5) multiplied by $\partial\alpha_1/\partial x$ to following forms, respectively,

$$(p_2-p_i)\frac{\partial\alpha_2}{\partial x}=L_2\left(\frac{\partial\alpha_2}{\partial x}-\frac{R_2}{2}\frac{\partial a_i}{\partial x}\right) \quad (8)$$

and

$$(p_i-p_1)\frac{\partial\alpha_1}{\partial x}=L_1\left(\frac{\partial\alpha_1}{\partial x}-\frac{R_1}{2}\frac{\partial a_i}{\partial x}\right). \quad (9)$$

Using the identity $\partial p_2/\partial x=\partial p_1/\partial x$, obtained from equation (3) for constant σ and δ, and the definition of acoustic speed, $C_k^2=\partial p_k/\partial\rho_k$, we can rewrite the conservation equations as

mass:

$$\rho_k\frac{\partial\alpha_k}{\partial t}+\frac{\alpha_k}{C_k^2}\frac{\partial p_2}{\partial x}+\rho_k v_k\frac{\partial\alpha_k}{\partial x}+\frac{\alpha_k v_k}{C_k^2}\frac{\partial p_2}{\partial x}+\rho_k\alpha_k\frac{\partial v_k}{\partial x}=0 \quad (10)$$

momentum:

$$\alpha_k\rho_k\frac{\partial v_k}{\partial t}+\alpha_k\frac{\partial p_2}{\partial x}+\alpha_k\rho_k v_k\frac{\partial v_k}{\partial x}+L_k\frac{\partial\alpha_k}{\partial x}=(-1)^n L_k\frac{R_k}{2}\frac{\partial a_i}{\partial x} \quad (11)$$

where n is the exponent denoting liquid for $n=1$ and gas for $n=2$ for the model of a gas bubble in the liquid. The above partial differential equations finally take a compact matrix form

$$\mathbf{A}(\mathbf{H})\frac{\partial \mathbf{H}}{\partial t}+\mathbf{B}(\mathbf{H})\frac{\partial \mathbf{H}}{\partial x}=\mathbf{E}(\mathbf{H}) \quad (12)$$

where $\mathbf{H}$ is the state vector made of the four primitive variables; α_2, p_2 and flow speeds v_1, v_2. The eigenvalues of the coefficient matrix $\mathbf{G}=\mathbf{A}^{-1}\cdot\mathbf{B}$ of equation (12) are determined from the characteristic equation, $Det(\mathbf{G}-\lambda\mathbf{I})=0$. According to Courant and Hilbert [9], the matrix $\mathbf{G}$ is hyperbolic at $\mathbf{H}$ if and only if $\mathbf{G}$ has a set of characteristic values with elements all real and a set of characteristic vectors that is complete. This case holds true when the derivative $\partial a_i/\partial x$ in equation (11) can be expressed as a function of the state vector $\mathbf{H}$, which is normally practice in the experimental two-phase flow studies.

We will now show that the characteristic equation has three complete sets of four real eigenvalues which depend on the effective bulk moduli L_1 and L_2. It is then shown that the mathematical ill-posedness of the two-fluids flow formulation is, indeed, eliminated by the concept of surface tension acting on a 'thin film'.

The effective bulk moduli, L_1 and L_2, can be obtained from the simplified physical models as follows. First, for the slug flow regime, there is no elastic interaction between the two fluids while the acoustic wave traveling in one fluid is not disturbed by the other fluid. The total time taken by the acoustic wave to travel in a column of slug-flows is then equal to the sum of the propagation time taken in each phase of the slug column made of a single phase. In other words, the effective bulk modulus of each phase is no longer different from that of the single phase:

$$L_1=\rho_1 C_1^2 \quad (13)$$

$$L_2=\rho_2 C_2^2. \quad (14)$$

Mixture bulk modulus of the two fluids takes, on the other hand, the combination form

$$L_m=-V\frac{\mathrm{d}p}{\mathrm{d}V}=-V\frac{\mathrm{d}p}{\mathrm{d}V_1+\mathrm{d}V_2}=V\frac{\mathrm{d}p}{\dfrac{V_1\,\mathrm{d}p}{L_{1,s}}+\dfrac{V_2\,\mathrm{d}p}{L_{2,s}}} \quad (15)$$

where $L_{1,s}$ and $L_{2,s}$ are the bulk moduli of the single phases. For the liquid–gas two phases, it clearly holds that $L_{2,s}\ll L_{1,s}$. Then, the above equation indicates that the mixture bulk modulus is closer to that of the gas than that of the liquid.

For the homogeneous two-fluids flow, assuming that the two fluids have the same averaged effective bulk modulus, we can approximately take the effective bulk moduli as

$$L_1=\rho_2 C_2^2 \quad (16)$$

$$L_2=\rho_2 C_2^2. \quad (17)$$

In case of the separated two-fluids flow, it is known that the pressure wave in a gas is not transmitted into the liquid but most of it is reflected; otherwise, the wave could change into capillary waves on the liquid surface. Unfortunately, detailed mechanism of pressure wave propagation into the parallel interface has not been well analyzed yet. We set here the values of L_k approximately as follows:

$$L_1=0 \quad (18)$$

$$L_2=\rho_2 C_2^2. \quad (19)$$

The eigenvalue λ_1 is the effective acoustic speed in the liquid and λ_2 is that in the gas phase, both influenced by the elasticity of the two fluids. They are distinguished from the acoustic speed in the single phase. The above eigenvalues are in good agreement with the experimental data [10] and show values nearly identical to the theoretical result [11] based on the elastic theory; see Table 1 and Fig. 2.

3. Surface tension effect in the two-fluids equation system

We will show that the eigenvalues convey significant physical meaning in the parameter plane, even outside of the three physical regimes shown in Table 1. The fourth-order characteristic equation can be written as

$$p_4(\lambda)=(\lambda-v_1)^2(\lambda-v_2)^2-K_1(\lambda-v_2)^2-K_2(\lambda-v_1)^2+K_3=0 \quad (20)$$

where,

$$K_1=\frac{L_1\alpha_2 C_1^2+C_2^2C_1^2\alpha_1\rho_2}{\alpha_2\rho_1 C_1^2+\alpha_1\rho_2 C_2^2}$$

$$K_2=\frac{L_2\alpha_1 C_2^2+C_1^2C_2^2\alpha_2\rho_1}{\alpha_2\rho_1 C_1^2+\alpha_1\rho_2 C_2^2}$$

and

$$K_3=\frac{C_2^2C_1^2(\alpha_2 L_1+\alpha_1 L_2)}{\alpha_2\rho_1 C_1^2+\alpha_1\rho_2 C_2^2}.$$

Other than the three sets of four real eigenvalues shown in Table 1, we could not obtain analytic form of the roots. However, numerical calculation can be performed point by point to check whether the roots are real or complex. For example, we set relatively a violent mixing mode by taking $\alpha_1=\alpha_2=0.5$, $v_1=10$, $v_2=20$ m s^{-1} for saturated water–vapor at the atmospheric pressure. In the parameter plane of Fig. 3(a) and (b), the marker '●', '○', and '■' stands, respectively, for the case of four real roots, two reals and a pair of complex, and two reals and a double root. No real roots could be found in the rest of the plane. Figure 3 (b)–(d) show small regions sequentially zoomed-up from Fig. 3(a).

Table 1
The acoustic speed in each phase of physical flow regimes

Flow regime	Models: Present eigenvalues	Models: Nugyen results
Homogeneous flow	$\lambda_1 = \pm C_1\sqrt{\dfrac{\rho_2 C_2^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$ $\lambda_2 = \pm C_2$	$\lambda = \pm C_1\sqrt{\dfrac{\rho_2 C_2^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$ $\lambda_2 = \pm C_2\sqrt{\dfrac{\rho_1 C_1^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$
Slug flow	$\lambda_1 = \pm C_1$ $\lambda_2 = \pm C_2$	$\lambda_1 = \pm C_1$ $\lambda_2 = \pm C_2$
Separated flow	$\lambda_1 = \pm C_1\sqrt{\dfrac{\alpha_1\rho_2 C_2^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$ $\lambda_2 = \pm C_2$	$\lambda_1 = \pm C_1\sqrt{\dfrac{\alpha_1\rho_2 C_2^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$ $\lambda_2 = \pm C_2\sqrt{\dfrac{\alpha_2\rho_1 C_1^2}{\alpha_1\rho_2 C_2^2+\alpha_2\rho_1 C_1^2}}$

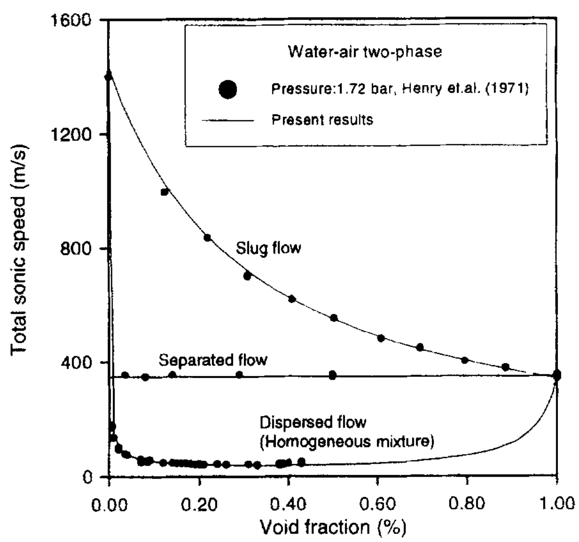

Fig. 2. Total acoustic speed is given by $\lambda_1\lambda_2/(\alpha_2\lambda_1+\alpha_1\lambda_2)$ for the slug and homogeneous flows. For a vertical pipe, it is measured at the bottom while a speaker is placed at the top. For a separated flow in the horizontal pipe, it is measured only in the air side and compared with λ_2 in Table 1.

The root pattern near the origin, for $L_1 \sim 0$ and $L_2 \sim 0$, is especially noteworthy. The characteristic polynomial, $P_4(\lambda)$, can be simplified by the Galilean transformation $v_1 \to w$, $v_2 \to -w$, where $w = (v_2 - v_1)/2$. Then, P_4 is reduced to

$$P_4(\lambda) = (\lambda^2 - w^2)^2 - K_1(\lambda + w)^2 - K_2(\lambda - w)^2 + K_3. \tag{21}$$

With the condition $L_1 \sim 0$ and $L_2 \sim 0$, the above equation is further reduced to

$$P_4(\lambda) \cong \lambda^4 + a\lambda^2 + b\lambda + c \tag{22}$$

where

$$\xi = \frac{1}{2}(K_1 + K_2) \cong \frac{1}{2}\frac{C_2^2 C_1^2(\alpha_1\rho_2 + \alpha_2\rho_1)}{\alpha_2\rho_1 C_1^2 + \alpha_1\rho_2 C_2^2}$$

$$\zeta = \frac{1}{2}(K_2 - K_1) \cong \frac{1}{2}\frac{C_2^2 C_1^2(\alpha_2\rho_1 - \alpha_1\rho_2)}{\alpha_2\rho_1 C_1^2 + \alpha_1\rho_2 C_2^2}$$

$$a = -2(w^2 + \xi)$$

$$b = -4w\zeta$$

and

$$c = w^2(w^2 - w\xi).$$

Since the relative velocity, $w = v_2 - v_1$, is very small when compared with the acoustic velocities, C_1 and C_2, for most cases of two-phase flow, the above polynomial becomes symmetric about $w = 0$ as

$$P_4(\lambda) \cong \lambda^2(\lambda^2 - 2\xi). \tag{23}$$

This polynomial has "W" shape curve as shown in Fig. 4. Because this curve will be shifted a little by slight change in the values of L_1 and L_2, a variety of roots (complex, double, or two reals) can appear near the origin, $\lambda = 0$, in Fig. 4. This result clearly shows recursive pattern of roots as shown in Fig. 3(c) and (d), which are sequentially zoomed-up from a small region. It reminds us of the fractal-like patterns of stars in the cosmos. Here, the markers '■', '●', and '•' in Fig. 3(c) and (d), the denotation of which is changed for plotting convenience, are equal to the markers '•', '○', and '■', respectively,

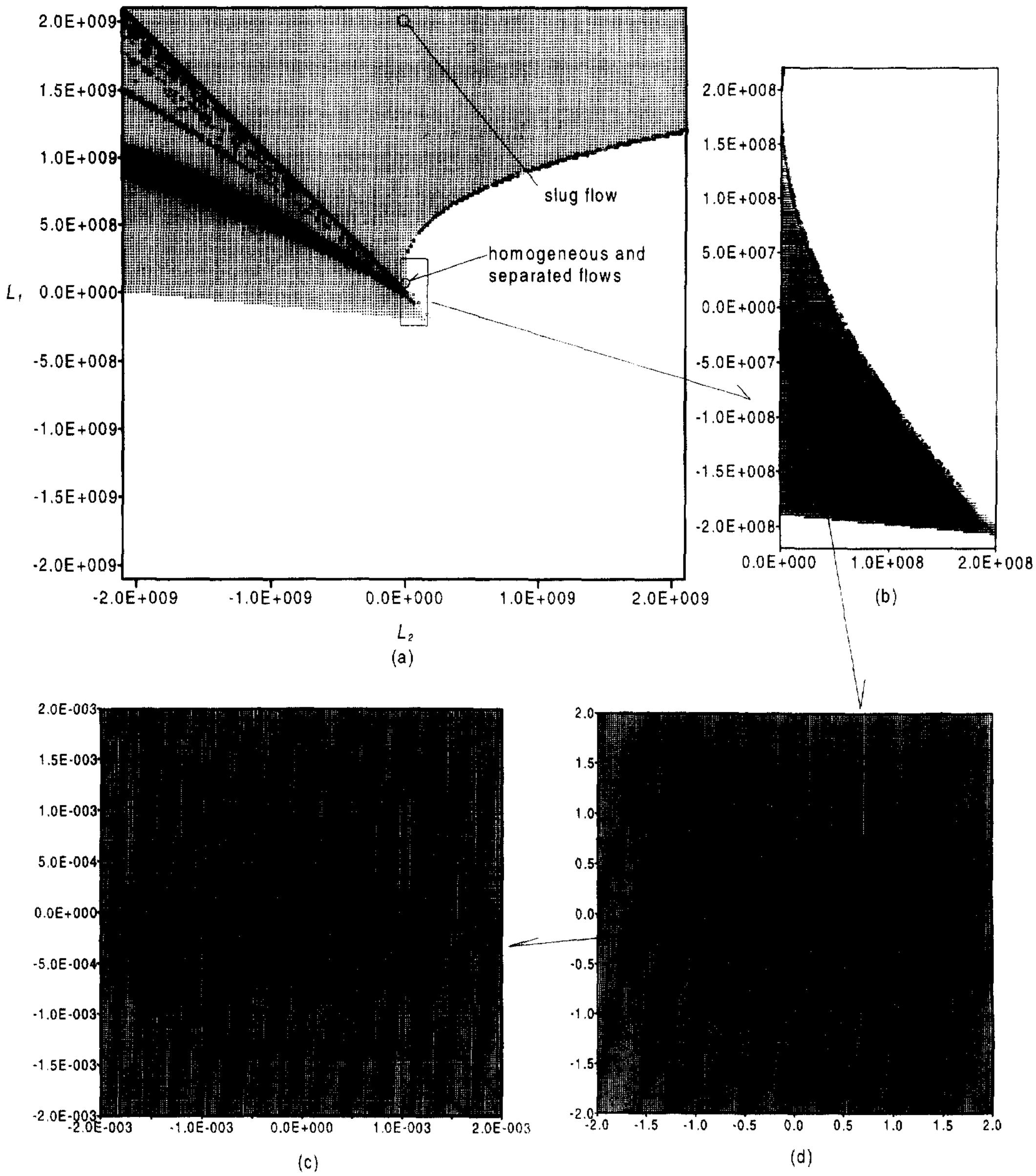

Fig. 3. (a) Parameter plane showing nature of the roots; (b), (c) and (d) are zoomed-up maps of small regions.

in Fig. 3(a) and (b). From this result, one can clearly see why the classical equation system of the two-fluids model has so much instability, even by a very slight disturbance near the origin where the surface tension is assumed negligibly small. Only positive values of the effective bulk moduli L_1 and L_2 are physically allowed. In Fig. 3(b), the point $(L_1, L_2) = (1.9 \times 10^8, 0)$ then forms a transition point bridging the homogeneous or separated flow regime and the slug flow regime for the given void fraction, since there is no other passage than this particular point. The two effective acoustic speeds, λ_1 and λ_2, take almost identical values at this point, about 455 m s^{-1}

4. Conclusion

One of the major causes of ill-posedness in the two-fluids equation system has been eliminated in the present

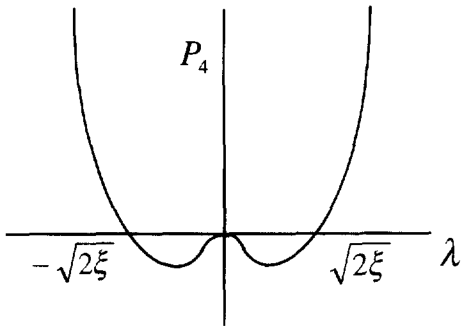

Fig. 4. The behavior of characteristic polynomial for negligibly small surface region.

paper by introducing a new concept called 'surface tension thickness'. It proved that the weakly well-posed equation system derived from this new concept, 'weakly' in the sense that the source vector **E** of equation (12) still can have derivative of unknowns, has three complete sets of real eigenvalues that depend on the values of effective bulk moduli, L_1 and L_2. When the surface tension effect is negligibly small, these bulk moduli lead to the eigenvalue pattern very sensitive to the small disturbances and having similarity in the closed-up maps. The existing physical flow regimes in the two-fluids flow such as the homogeneous, slug, and separated flows are well represented by the three sets of real eigenvalues obtained in this paper. The acoustic speeds in the two-phase flow predicted by the present formulation showing excellent agreement with the experimental data suggest that the present theory can be further developed to explain the multi-dimensional two-phase flows.

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